

**The Factors of  $(a, b, c, f, g, h)(x, y, z)^2 - \lambda(x^2 + y^2 + z^2)$ .**

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If  $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ ,  
and  $S \equiv x^2 + y^2 + z^2$ ,

$f - \lambda S$  is the product of two factors of the form  $ax + \beta y + \gamma z$  if  $\lambda$  is a root of a discriminating cubic

$$\begin{vmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{vmatrix} = 0.$$

A well-known proof of the reality of the roots of the cubic is as follows :—

Write

$$\phi(\lambda) \equiv (\lambda - a)\{(\lambda - b)(\lambda - c) - f^2\} - \{(\lambda - b)g^2 + (\lambda - c)h^2 + 2fgh\},$$

and  $\psi(\lambda) \equiv (\lambda - b)(\lambda - c) - f^2$ .

Suppose that  $a > b > c$ ;

when  $\lambda = +\infty, \quad b, \quad c, \quad -\infty,$

$$\psi(\lambda) = +\infty, \quad -f^2, \quad -c^2, \quad +\infty.$$

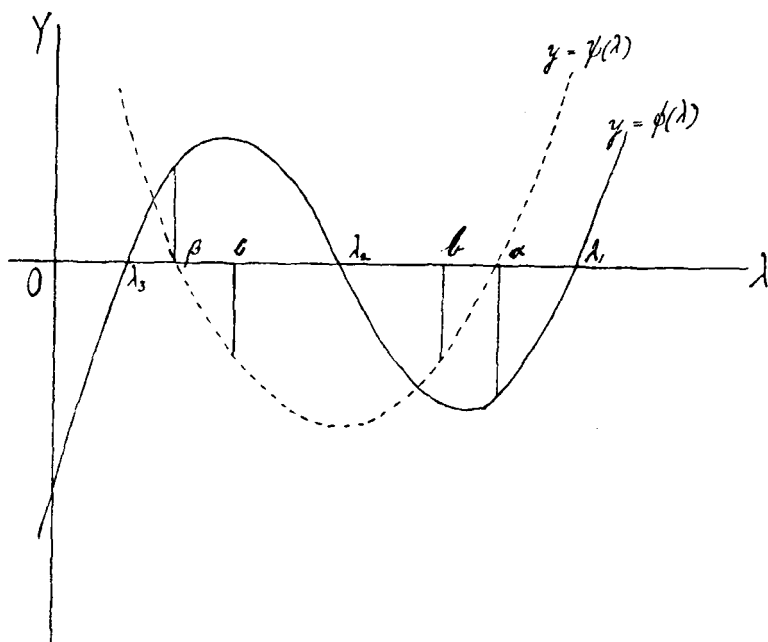
Hence, (see figure), the equation  $\psi(\lambda) = 0$  has two real roots,  $a$  and  $\beta$ , such that

$$a > b > c > \beta.$$

When

$$\begin{array}{ccccccc} \lambda = +\infty, & a, & & \beta, & & -\infty, \\ \phi(\lambda) = +\infty, & -(\sqrt{a-b}g \pm \sqrt{a-ch})^2, & (\sqrt{b-\beta}g \pm \sqrt{c-\beta}h)^2, & -\infty. \end{array}$$

Hence the cubic,  $\phi(\lambda) = 0$ , has three real roots,  $\lambda_1, \lambda_2, \lambda_3$ , such that  $\lambda_1 > a > \lambda_2 > \beta > \lambda_3$ .



Now

$$f - \lambda S \equiv \frac{1}{b - \lambda} \left[ \{hx + (b - \lambda)y + fz\}^2 + \frac{1}{\psi(\lambda)} \{z\psi(\lambda) - x(hf - \overline{b - \lambda}g)\}^2 \right].$$

Therefore if  $\lambda = \lambda_1$ ,  $b - \lambda < 0$  and  $\psi(\lambda) > 0$ , and  $f - \lambda S$  is of the form  $-(u^2 + v^2)$ , where  $u$  and  $v$  are linear functions of  $x, y, z$ , with real coefficients. If  $\lambda = \lambda_2$ ,  $b - \lambda \leq 0$ , and  $\psi(\lambda) < 0$ , and  $f - \lambda S$  is of the form  $\pm(u^2 - v^2)$ . If  $\lambda = \lambda_3$ ,  $b - \lambda > 0$ , and  $\psi(\lambda) > 0$ , and  $f - \lambda S$  is of the form  $u^2 + v^2$ .

The only value of  $\lambda$  for which  $f - \lambda S$  is the product of factors with real coefficients is therefore the mean value  $\lambda_2$ .

The result can be applied to find the real circular sections of the conicoid  $f(x, y, z) = 1$ . Write the equation

$$f(x, y, z) - \lambda(x^2 + y^2 + z^2) + \lambda(x^2 + y^2 + z^2) - 1 = 0,$$

and it appears that if  $f - \lambda S = 0$  represents a pair of planes, the planes cut the conicoid in circles. The real circular sections are given by the mean root of the discriminating cubic.